SOLUTIONS

SECTION 5.3: INTEGRAL TEST AND p-SERIES

1. The Integral Test:

if $f(x)$ is decreasing and continuous, and $f(x) \geq 0$, and if $f(n) = a_n$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_N^{\infty} f(x)\,dx$$

either both converge or both diverge

2. All questions below refer to the series $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{10+n^2}$

(a) What does the Divergence Test tell us about this series?

$a_n = \frac{3n}{10+n^2}$ so $\lim_{n\to\infty} \frac{3n}{10+n^2} \overset{LH}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{n\to\infty} \frac{3}{2n} = 0$ $\therefore$ nothing (inconclusive)

(b) Show that we can apply the Integral Test to the series.

$f(x) = \frac{3x}{10+x^2}$ is continuous and (eventually) decreasing

(c) Use the Integral Test to determine whether or not the series converges.

$$\int_0^{\infty} \frac{3x}{10+x^2}\,dx = \int_{10}^{\infty} \frac{3 \cdot \frac{du}{2}}{u} = \frac{3}{2}\int_{10}^{\infty} \frac{du}{u}$$

$u = 10 + x^2$, $\frac{du}{2} = x\,dx$

$$= \lim_{t\to\infty} \frac{3}{2} \ln|u| \Big]_{10}^{t} = \frac{3}{2} \lim_{t\to\infty} (\ln t - \ln 10) = +\infty$$

$\lim_{t\to\infty} \ln t = +\infty$

$\therefore$ series diverges

3. A **p-series** has the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

4. Convergence: A p-series converges when $p>1$ ← *give a condition on p*
and diverges when $p \le 1$ ← *give a condition on p*

↖ harmonic with $p=1$ diverges

5. Use what we know about p-series and convergence to determine whether the series below converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{1.56}}$ $\quad p = 1.56 > 1 \therefore$ converges

(b) $\sum_{n=1}^{\infty} \frac{1}{n^{99/100}}$ $\quad p = \frac{99}{100} = .99 \therefore$ diverges

6. Use the integral test or divergence test to determine whether the series converge:

(a) $\sum_{n=1}^{\infty} \frac{n}{3^n}$ integral test: $\int_1^{\infty} x 3^{-x}\,dx$ } IBP: $u=x$ / $v = \frac{-1}{\ln 3}3^{-x}$, $du=dx$ / $dv = 3^{-x}dx$

$$= \lim_{t\to\infty}\left[\frac{-1}{\ln 3} x 3^{-x}\right]_1^t + \frac{1}{\ln 3}\int_1^{\infty} 3^{-x}\,dx$$

$$= 0 + \frac{1}{\ln 3}\frac{1}{3} + \frac{1}{\ln 3}\lim_{t\to\infty}\left[\frac{-1}{\ln 3}3^{-x}\right]_1^t = \frac{1}{3\ln 3} + \frac{1}{\ln 3}\left(0 + \frac{1}{\ln 3}\right) < \infty$$

$\therefore$ converges

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ integral test

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx \overset{u=\ln x}{\underset{du=\frac{dx}{x}}{=}} \int_{\ln 2}^{\infty} \frac{du}{u^2} = \lim_{t\to\infty}\left[-u^{-1}\right]_{\ln 2}^{t}$$

$$= -0 + \frac{1}{\ln 2} < \infty \therefore \text{converges}$$

(c) $\sum_{n=2}^{\infty} \frac{n}{\ln n}$

divergence test $\quad a_n = \frac{n}{\ln n}$

$$\lim_{n\to\infty}\frac{n}{\ln n} \overset{LH}{\underset{\frac{\infty}{\infty}}{=}} \lim_{n\to\infty}\frac{1}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} n = +\infty \neq 0$$

$\therefore$ diverges