

Lecture Notes: Arrow's Impossibility Theorem

Math F113X Math and Society

What can go wrong?

Complete the Borda Method on the preference schedule below. Is there anything that bothers you about the winner?

pts		21	19
3	1st	A	B
2	2nd	B	C
1	3rd	C	A

$$A: 3(21) + 1(19) = 82$$

$$B: 2(21) + 3(19) = 99 \leftarrow B \text{ wins in Borda Count!}$$

$$C: 1(21) + 2(19) = 59$$

$$\# \text{ voters} = 21 + 19 = 40, \quad \frac{40}{2} = 20$$

$$\text{majority} \geq 21 \leftarrow \text{Candidate A has a majority!}$$

→ **Majority Criterion**

If a candidate receives a majority of the votes, they should win the election

What can go wrong?

(Example 11) Below is a preference schedule (left) and the results from all one-to-one matchups (right). What happens if we remove D ?

	5	5	6	4
1st	D	A	C	B
2nd	A	C	B	D
3rd	C	B	D	A
4th	B	D	A	C

Copeland Tally

A: $\frac{1}{2} + 1 = 1.5$

B: $1 + \frac{1}{2}$

C: 2 ← winner!

D: 1

Suppose D wasn't there?

Copeland Tally

A = $\frac{1}{2} + 1 = 1.5$

B = $\frac{1}{2}$ ↑ Winner!

C = 1

matchup	A vs. B	A vs. C	A vs. D	B vs. C	B vs. D	C vs. D
tally	A : 10 B : 10	A : 14 C : 6	A : 5 D : 15	B : 4 C : 16	B : 15 D : 5	C : 11 D : 9
winner	tie	A	D	C	B	C

What can go wrong?

Independence of irrelevant alternatives

IIA

Can be phrased in 2 ways:

- ① If you eliminate a non-winning candidate from the ballot, it shouldn't change the outcome of the election!
- ② If candidate X is preferred over candidate Y that shouldn't change just because we introduce candidate Z .

Recap of Fairness Conditions

Condition	Pass	Fail
IIA		Copeland, Borda, IRV, Plurality
Majority	IRV, Plurality Copeland	Borda
Monotonicity	Copeland, Plurality Borda	IRV
Condorcet	Copeland*	Plurality, IRV, Borda

Criterion \ Method		Majority winner	Majority loser	Mutual majority	Condorcet winner [Tn 1]	Condorcet loser	Pareto	Smith	Smith-IIA	IIA/LIIA [Tn 1]	IPDA	Cloneproof	Monotone	Consistency [Tn 1]	Participation [Tn 1]	Reversal symmetry	Homogeneity	Later-no-harm [Tn 1]	Later-no-help [Tn 1]	Sincere favorite [Tn 1]
Plurality		Yes	No	No	No	No	Yes	No	No	No		No	Yes	Yes	Yes	No	Yes	Yes	Yes	No
Anti-plurality		No	Yes	No	No	No	No	No	No	No	No	No	Yes	Yes	Yes	No	Yes	No	No	Yes
Two round system		Yes	Yes	No	No	Yes		No	No	No		No	No	No	No		Yes	Yes	Yes	No
Random ballot [Tn 7]		No	No	No	No	No	Yes	No	Yes	Yes	Yes	Yes	Yes		Yes	No	Yes	Yes	Yes	Yes
Maximal lotteries		Yes			Yes	Yes	Yes	Yes		No		Yes	No	Yes	No					
Sortition [Tn 8]		No	No	No	No	No	No	No	Yes	Yes	No	No	Yes	N/A	Yes	N/A	N/A	Yes	Yes	Yes
Instant-runoff		Yes	Yes	Yes	No	Yes	Yes	No	No	No		Yes	No	No	No	No	Yes	Yes	Yes	No
Coombs		Yes	Yes	Yes	No	Yes		No	No	No	No	No	No	No	No		Yes	No	No	Yes
Nanson		Yes	Yes	Yes	Yes	Yes		Yes	No	No	No	No	No	No	No	Yes		No	No	No
Baldwin		Yes	Yes	Yes	Yes	Yes		Yes	No	No	No	No	No	No	No	No		No	No	No
Tideman alternative		Yes	Yes	Yes	Yes	Yes		Yes	Yes	No		Yes	No	No	No	No		No	No	No
Minimax		Yes	No	No	Yes [Tn 2]	No	Yes	No	No	No	Yes	No	Yes	No	No	No	Yes	No [Tn 2]	No	No
Copeland		Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	No	No	Yes	No	No			No	No	No
Black		Yes	Yes	No	Yes	Yes	Yes	No	No	No	No	No	Yes	No	No	Yes	Yes	No	No	No
Kemeny		Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	LIIA	No	No	Yes	No	No	Yes	Yes	No	No	No
Ranked pairs		Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	LIIA	No	Yes	Yes	No	No [Tn 3]	Yes	Yes	No	No	No
Schulze		Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	No	Yes	Yes	No	No [Tn 3]	Yes	Yes	No	No	No
Borda		No	Yes	No	No	Yes	Yes	No	No	No	No	No	Yes	Yes	Yes	Yes	Yes	No	Yes	No
Bucklin		Yes	Yes	Yes	No	No		No	No	No	No	No	Yes	No	No	No	Yes	No	Yes	No
Dodgson		Yes	No	No	Yes	No		No	No	No		No	No	No	No	No	No	No	No	No
Approval		Yes	No	No	No	No	Yes	No	Yes	Yes [Tn 4]	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes	Yes
Majority Judgement		No	No [Tn 5]	No [Tn 6]	No	No		No	Yes	Yes [Tn 4]		Yes	Yes	No	No [Tn 3]		Yes	No	Yes	Yes
Score		No	No	No	No	No		No	Yes	Yes [Tn 4]		Yes	Yes	Yes	Yes	Yes	Yes	No	Yes	Yes
STAR		No	Yes	No	No	Yes		No	No	No	No	No	Yes	No	No	No		No	No	No
Quadratic		No	No	No	No	No		No	No	No		No	Yes	Yes	Yes			N/A	N/A	No

Arrow's Impossibility Theorem

You can't have a voting method
that satisfies ALL fairness
criteria simultaneously!

Approval Voting

	30	15	10	10
Anchorage	X	X		
Bettles	X		X	X
Chevak		X		X

Voters vote for
all the choices they
approve of!

Count total # of approvals

$$A = 30 + 15 = 45$$

$$B = 30 + 10 + 10 = 50 \text{ ← winner!}$$

$$C = 15 + 10 = 25$$

* Vulnerable to insincere voting!
If the 30 voters in col #1 only vote for A
they guarantee A wins