

Lecture Notes: Weighted Voting, Banzhaf Power Index

Math F113X Math and Society

Review

Answer questions about the weighted voting system below.

$[67 : 50, 30, 10, 5, 4, 1]$.
P₁ P₂ P₃ P₄ P₅ P₆

QUOTA

Weights =
of votes for
each player

(a) How many players are there? 6

(b) What is the quota? 67

Quota: need
 $\frac{2}{3}$ of votes!

(c) What is the total weight of the voting system? 100

$$50 + 30 + 10 + 5 + 4 + 1 = 100$$

Review: Terminology

Explain the terminology below. Then use the voting system from #1 to add illuminating examples.

$$[67 : 50, 30, \overset{P_3}{10}, \overset{P_4}{5}, \overset{P_5}{4}, \underline{1}]$$

(a) coalition

A group of players that vote together

$$\text{Ex: } \{P_3, P_4, P_5\}$$
$$\text{weight} = 10 + 5 + 4 = 19 < 67$$

(b) winning coalition

A coalition whose weights sum to at least the quota

$$\underline{\text{Ex: } \{P_1, P_2\}}$$
$$\text{weight} = 50 + 30 = \underline{80} \geq \underline{67}$$

Winning coalition!

Review: Terminology

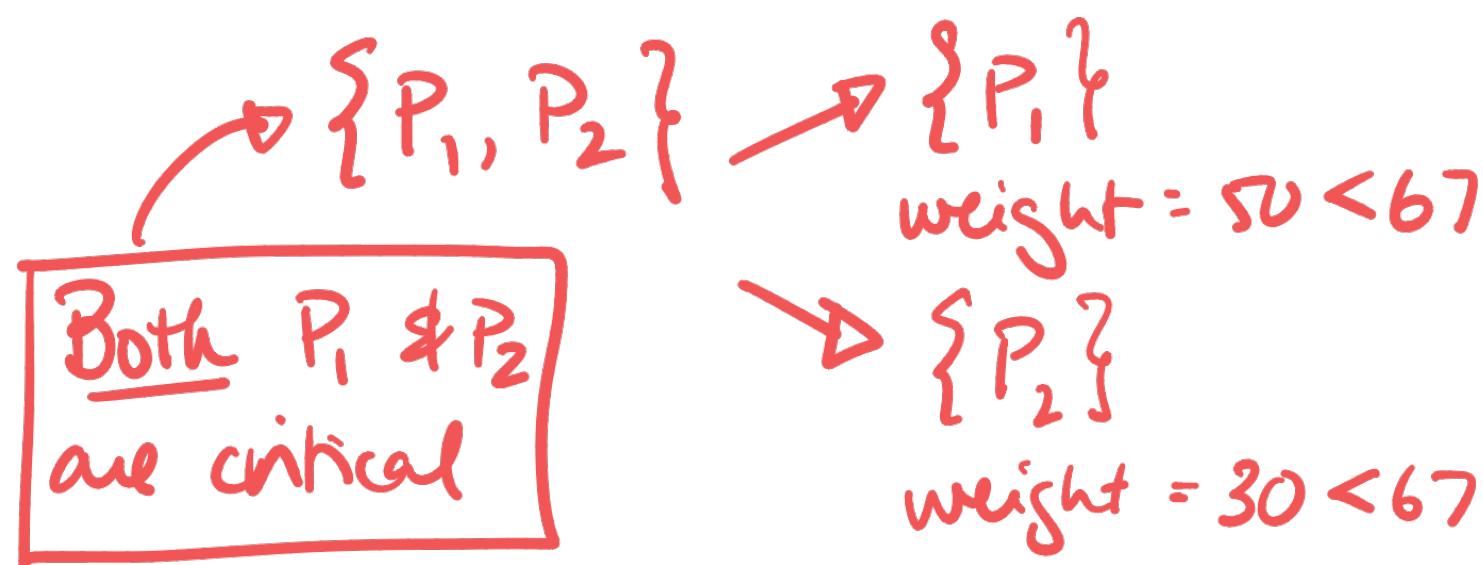
(c) critical player

Removal of this player
moves coalition from
winning to losing

(d) dictator

Player who can make
quota alone!

$[67 : \overset{P_1}{50}, \overset{P_2}{30}, 10, 5, 4, 1]$



But in $\{P_1, P_2, P_3\}$, P_3 is
NOT critical, since $\{P_1, P_2\}$
is still winning!

Review: Terminology

(e) a player with veto power

→ In every winning coalition
→ Sum of all other players
 $< \text{quota}$

(f) a dummy player

Never critical

$[67 : 50, 30, 10, 5, 4, \underline{1}]$

$\{P_2, P_3, P_4, P_5, P_6\}$

weight = $30 + 10 + 5 + 4 + 1$
 $= 50 < 67$

* Any winning coalition
 must include P_1

P_6 is
a dummy

If P_6 were critical, need
coalition s.t. removing P_6
went win $\rightarrow$ lose
But no coalition w/ weight $66 = 67 - 1$

Coalitions in a Weighted Voting System

Answer questions about the weighted voting system below.

$[30 : \overset{P_1}{25}, \overset{P_2}{10}, \overset{P_3}{5}]$

(a) List **ALL** coalitions.

one player	Coalition weight	2 players	Coalition weight	3 players	Coalition weight
P_1	25	P_1, P_2	$25 + 10 = 35$	P_1, P_2, P_3	$25 + 10 + 5 = 40$
P_2	10	P_1, P_3	$25 + 5 = 30$		
P_3	5	P_2, P_3	$10 + 5 = 15$		

Coalitions in a Weighted Voting System

[30 : 25, 10, 5]

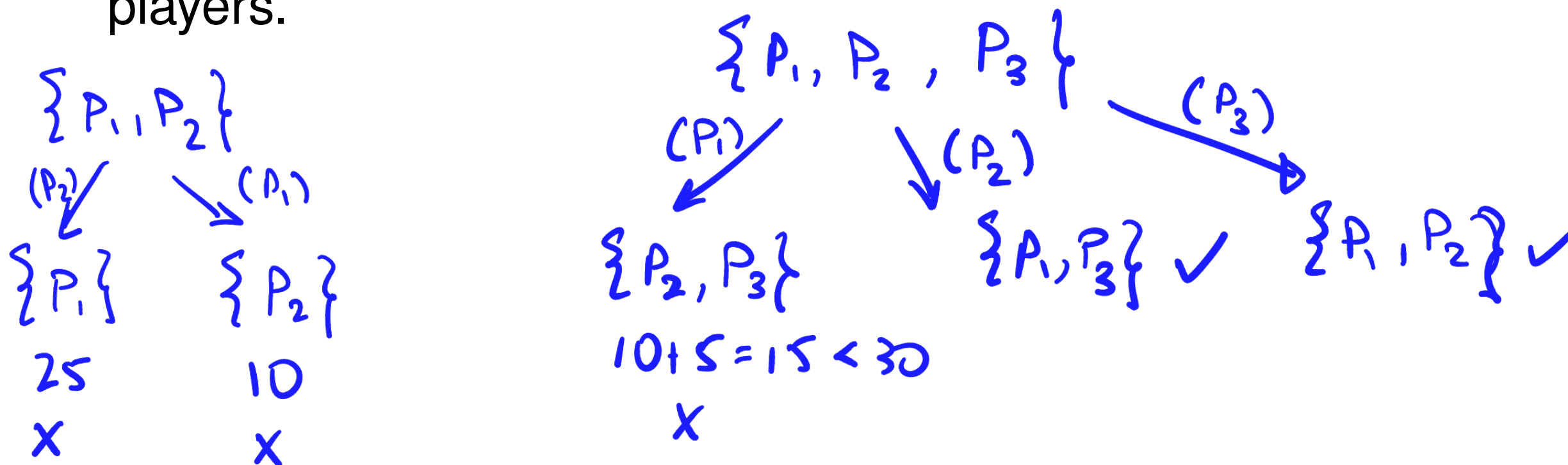
(b) List all **WINNING** coalitions.

$$\{\underline{P_1}, \underline{P_2}\}$$

$$\{\underline{P_1}, \underline{P_3}\}$$

$$\{\underline{P_1}, P_2, P_3\}$$

(c) In each **winning** coalition listed above, underline the critical players.



The Banzhaf Power Index

(d) Calculate the Banzhaf Power index.

- (i) Find all winning coalitions
- (ii) Find all critical players
- (iii) Underline critical players
- (iv) Count total # underlines
- (v) For each player, compute:

$$\frac{\text{\# times the player is underlined}}{\text{total \# underlines}}$$

$\{\underline{P_1}, \underline{P_2}\}$
 $\{\underline{P_1}, \underline{P_3}\}$
 $\{\underline{P_1}, P_2, P_3\}$

5 total underlines!

player	# underlines	$\frac{\text{\# times}}{\text{total}}$
P_1	3	$\frac{3}{5} = 60\%$
P_2	1	$\frac{1}{5} = 20\%$
P_3	1	$\frac{1}{5} = 20\%$

Banzhaf Power Index

sum to 100%