# Lecture Notes: Weighted Voting, Wrap-up

Math F113X Math and Society

# Counting Coalitions

In earlier notes, we found seven possible coalitions with players $P_1$, $P_2$, and $P_3$.

(a) List them again below.

$P_1$ $\quad$ $P_1, P_2$ $\quad$ $P_1, P_2, P_3$

$P_2$ $\quad$ $P_1, P_3$

$P_3$ $\quad$ $P_2, P_3$

# Counting Coalitions

(b) Suppose the system has a fourth player, $P_4$. Determine how many coalitions in this case. Try to answer the question without actually listing all of them.

$P_1$   $P_1, P_2$   $P_1, P_2, P_3$   $P_1, P_4$   $P_1, P_2, P_4$   $P_1, P_2, P_3, P_4$

$P_2$   $P_1, P_3$   $P_2, P_4$   $P_1, P_3, P_4$

$P_3$   $P_2, P_3$   $P_3, P_4$   $P_2, P_3, P_4$   $P_4$

(c) What if there is a fifth player, $P_5$?

Total $= 7 + 7 + 1$

$= 15$

guess: $15 + 15 + 1$

$= 31$

# Counting Coalitions

(d) Make a $\overbrace{\text{conjecture}}^{\text{guess}}$ about how many coalitions are possible with $n$ players, $P_1, P_2, P_3, \cdots P_n$. How would you argue that your count is correct?

Two players? $\{P_1\}, \{P_2\}, \{P_1 P_2\}$

pattern

| # players | # coalitions |
|---|---|
| 3 | $7 = 8 - 1$ |
| 4 | $15 = 16 - 1$ |
| 5 | $31 = 32 - 1$ |
| 2 | $3 = 4 - 1$ |
| 1 | $1 = 2 - 1$ |
| $n$ | $2^n - 1$ |

$2, 4, 8, 16, 32, \ldots$

$2^1, 2^2, 2^3, 2^4, 2^5, \ldots$

# Counting Coalitions

(e) What does this suggest about the mechanics of calculating the Banzhaf Power Index for a weighted voting system with a lot of players?

$2^5 = 32$ ← 31 possible coalitions with 5 players

To compute BPI:

write out all 31 coalitions

eliminate losing coalitions

determine critical players in each coalition

count underlines, compute BPI

$2^{10} = 1024$ $2^{20} > 1000000$

# What is the Banzhaf Power Index

(a) when there is a dictator

Dictator's power = 100%
everyone else = 0%

(b) for a dummy player

Always has 0%

# What is the Banzhaf Power Index

(c) if all players have an equal number of votes

All players have equal power!

(d) if player $P_1$ has double the number of votes as player $P_2$?

$P_2$ cannot have <u>more</u> power than $P_1$

$[11: 10, 5, 2]$

$\{\underline{P_1}, \underline{P_2}\}$

$\{\underline{P_1}, \underline{P_3}\}$

$\{\underline{P_1}, P_2, P_3\}$

| | | BPI |
|---|---|---|
| $P_1$ | 3 | 3/5 |
| $P_2$ | 1 | 1/5 |
| $P_3$ | 1 | 1/5 |

$[16: 10, 5, 2]$

$\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$

Equal power!

$[10: 10, 5, 2]$

$P_1$ is a dictator!

$P_1$ has 100% power

$P_2, P_3$ have 0% power