

Lecture Notes: Fair Division, Lone Divider

Math F113X Math and Society

Divider-Chooser with Two People

Tom and Fred were given a cake worth \$12 that is equal parts strawberry, vanilla and chocolate.

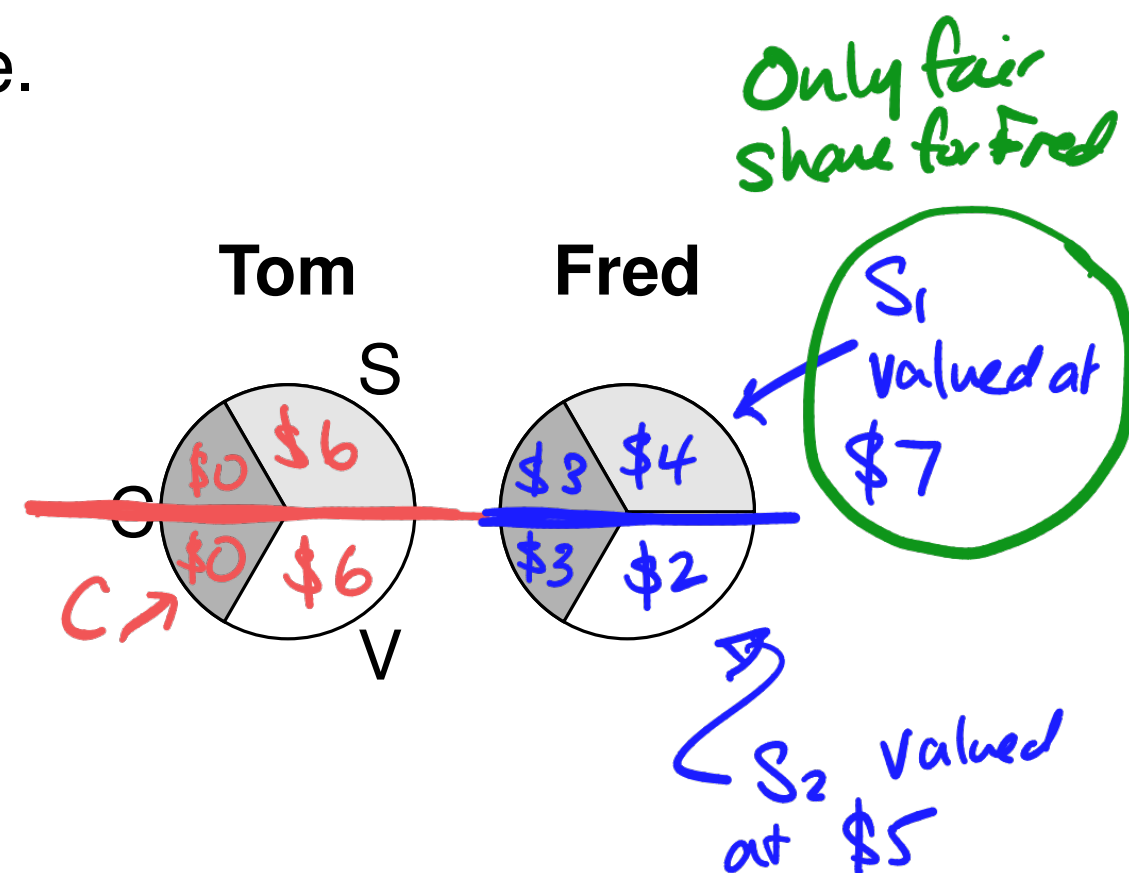
- (a) Divide the cake using Divider-Chooser assuming Tom is the divider. Determine the **value** of the assigned share to each party. Use the pictures of the cake to draw the outcome.

	vanilla	strawberry	chocolate
Tom	\$ 6	\$ 6	\$ 0
Fred	\$ 2	\$ 4	\$ 6

Tom's S_1 
 S_2 

FAIR SHARE:

$$\frac{\$12}{2} = \$6$$



What Happens with **Three** People?

(b) What happens if try to use Divider-Chooser with **three** people? (Try Fred as divider.)

FAIR
SHARE =
 $\frac{12}{3} = 4$

	V	S	C
Tom	\$ 6	\$ 6	\$ 0
Fred	\$ 2	<u>\$ 4</u>	\$ 6
Mary	\$ 8	\$ 0	<u>\$ 4</u>

Fred's share division:

$$S_1 = \frac{2}{3}C \quad S_2 = \frac{1}{3}C + V \quad S_3 = \text{all } S$$

Tom values
 $S_1 = \$0$

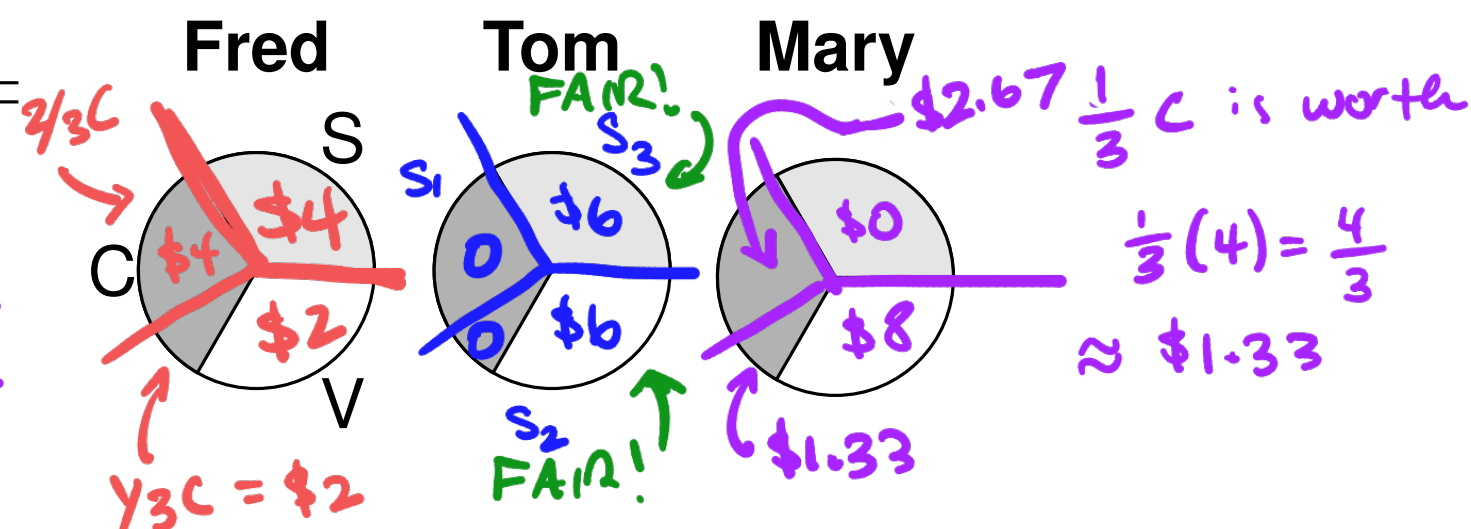
$$S_2 = \$6$$

$$S_3 = \$6$$

Mary values
 $S_1 = \$2.67$

$$S_2 = \$9.33$$

$$S_3 = \$0$$



Scenario #1: Tom chooses S_3

Tom gets S_3 (all strawberry)
worth \$6 to him

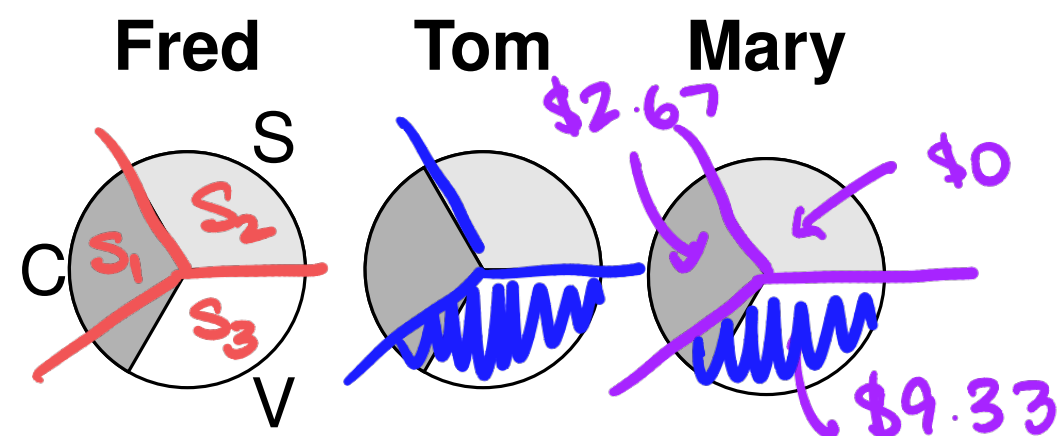
Mary chooses S_2 ($\frac{1}{3}C + V$) worth
\$9.33 to her

Fred gets S_1 ($\frac{2}{3}C$) worth \$6

All fair shares!

What Happens with **Three** People?

	V	S	C
Tom	\$ 6	\$ 6	\$ 0
Fred	\$ 2	\$ 4	\$ 6
Mary	\$ 8	\$ 0	\$ 4



Fred's share division:

$$S_1 = \frac{2}{3}C \quad S_2 = \frac{1}{3}C + V \quad S_3 = \text{all } S$$

Tom values

$$S_1 = \$0$$

$$S_2 = \$6$$

$$S_3 = \$6$$

Mary values

$$S_1 = \$2.67$$

$$S_2 = \$9.33$$

$$S_3 = \$0$$

Scenario # 2: Tom chooses S_2

Tom gets S_2 , worth \$6

Mary has NO AVAILABLE fair shares!

Divider-chooser can fail with more than 2 people!

Why Cake?

What about a car?

... or a house?

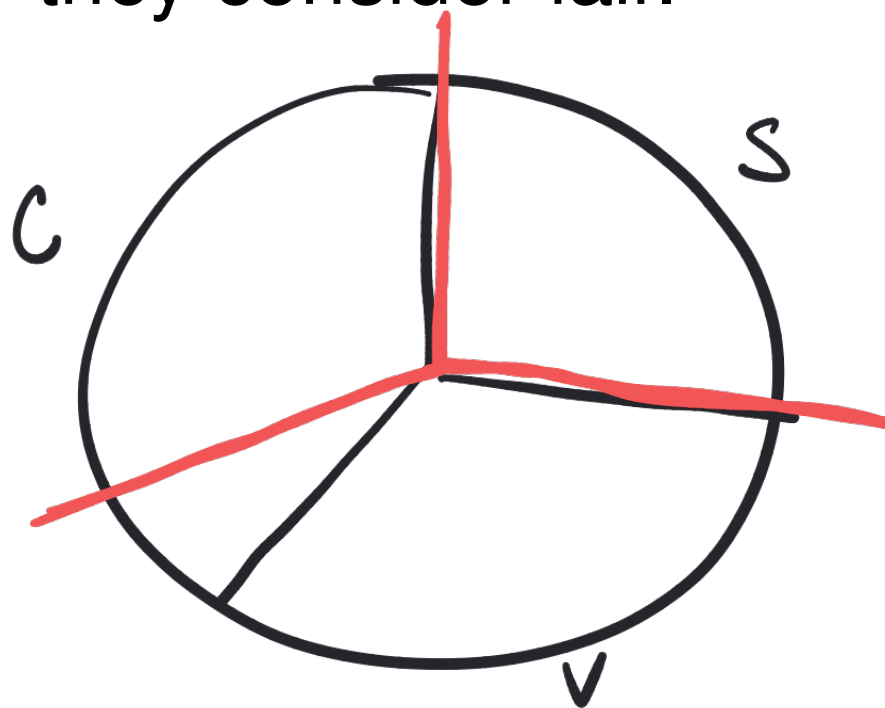
... or some furniture?

* We needed to be able to slice the cake into smaller pieces!

→ continuously divisible

The Lone-Divider Method (for N people with $N \geq 3$)

0. Arbitrarily pick a Divider.
1. Divider divides the items into N shares of equal value to them:
 $S_1, S_2, \dots, S_N$.
2. Remaining parties **declare** or **bid** on shares, $S_1, S_2, \dots, S_N$, they consider fair.



Divider = Fred

	$S_1 = \frac{2}{3}C$	$S_2 = \frac{1}{3}C + V$	$S_3 = S$
Fred	\$4	\$4	\$4
Tom	\$0	\$6	\$6
Mary	\$2.67	\$9.33	\$0

The Lone-Divider Method (for N people with $N \geq 3$)

3. (a) **IF** the N shares can be divided such that each all get a fair share, then do so.
- (b) **IF NOT**, the Divider gets a **non-contested piece**. Remaining shares are recombined. Lone-Divider is restarted with $N - 1$ parties.
- (c) If only 2 parties remain, use Divider-Chooser.

Divider = Fred

	$S_1 = \frac{2}{3}C$	$S_2 = \frac{1}{3}C + V$	$S_3 = S$
Fred	\$4	\$4	\$4
Tom	\$0	\$6	\$6
Mary	\$2.67	\$9.33	\$0

We can distribute
so that everyone is
happy!

Example 1 Football Memorabilia

Suppose Patrick, Chris, and Travis are splitting a pile of football memorabilia estimated to be worth \$300. It has been split into 3 shares and their respective values are summarized in the table.

	s_1	s_2	s_3
Patrick	\$50	\$150	\$100
Chris	\$70	\$70	\$160
Travis	\$100	\$100	\$100

Example 1 Football Memorabilia

(a) What is a fair share? \$100

Total Value = \$300

3 people $\Rightarrow$ fair share $\geq \frac{300}{3} = 100 \rightarrow$

(b) Circle or highlight each individual's **bid** (the shares they would consider to be fair).

(c) Determine which person was the Divider.

Travis, because he valued all the shares equally

	S ₁	S ₂	S ₃
Patrick	\$50	\$150	\$100
Chris	\$70	\$70	\$160
Travis	\$100	\$100	\$100

Example 1 Football Memorabilia

(d) Determine the next steps of the Lone-Divider Method.

person	bid	assign	\$	fair?
patrick	S_2, S_3	S_2	\$150	✓
Chris	S_3	S_3	\$160	✓
Travis	S_1, S_2, S_3	S_1	\$100	✓

	S_1	S_2	S_3
Patrick	\$50	\$150	\$100
Chris	\$70	\$70	\$160
Travis	\$100	\$100	\$100

Example 2 Football Memorabilia

Suppose Patrick, Chris, and Travis are splitting a pile of football memorabilia estimated to be worth \$300. It has been split into 3 (different) shares and their respective values are summarized in the table.

① Next Step: Patrick picks an uncontested share!
Let's say t_1

divider →

	t_1	t_2	t_3
Patrick	\$100	\$100	\$100
Chris	\$90	\$40	\$170
Travis	\$50	\$90	\$160

Contested

only 2 means do divider-chooser!

② Then recombine the rest of the shares! (t_2 & t_3)
stuff Now do Lone divider again