

Lecture Notes: Fair Division, Sealed Bids

Math F113X Math and Society

Sealed Bids: General Idea

Method to divide stuff that cannot be subdivided!

House

Furniture

Car

Assignment of value to items will be unequal

Exchange cash to make sure each person gets a fair share to them

Steps

- (a) List items to be divided
- (b) Individuals submit a sealed bid for each item
- (c) Determine each person's total bid
Add up their bids: this is what they value ALL the stuff at
- (d) Determine each person's fair share: $(\text{their total bid}) / (\# \text{ people})$
Fair share can be different for each person!
- (e) Award each item to the highest bidder
Each person has an award value.
- (f) Determine each person's initial allocation of money to **holding pot**:
 - if $(\text{award} - \text{fair share}) > 0$, individual pays difference **into** pot
 - if $(\text{award} - \text{fair share}) < 0$, individual gets difference **from** pot
- (g) Divide remaining **surplus in pot** equally

Example The Bids

ITEMS: Vase, Earrings, Ice Skates. PEOPLE: Sam, Omar *~ two people!*

	Sam	Omar
vase	\$10	\$5
earrings	\$4	\$15
ice skates	\$20	\$16
total bid	$10+4+20 = \$34$	$5+15+16 = \$36$
fair share	$\frac{34}{2} = \$17$	$\frac{36}{2} = \$18$

Example The Awards

	Sam	Omar
vase	\$10	\$5
earrings	\$4	\$15
ice skates	\$20	\$16

	Sam	Omar
<i>fair share (from above)</i>	\$17	\$18
who gets what (award)	Vase, skates	earrings
value of award	$10 + 20 = \$30$	\$15
award value — fair share	$30 - 17 = \$13$	$15 - 18 = -\$3$
pays in / receives	<u>PAYS</u> \$13	<u>GETS</u> \$3

Right now: Sam got 30, paid 13, total = 17
 Omar got 15, got 3, total = 18

$$\begin{array}{r}
 \text{POT} \\
 \hline
 +13 \\
 -3 \\
 \hline
 \$10 \leftarrow \text{Surplus!}
 \end{array}$$

Example The Surplus

	Sam	Omar
<i>pays in / receives (from above)</i>	Paid \$13	got \$3
total surplus	$13 - 3 = \$10$	
share of surplus	$\frac{10}{2} = \$5$	$\frac{10}{2} = \$5$
Final allocation	Vase, skates Paid \$13, got \$5 Surplus.	Earrings Paid \$3 (pot) \$5 (surplus)

Total PAY to Omar is \$8 Total from Sam is \$8

Example Summary of Final Allocation

person	items	cash	paid or received?
Sam	Vase Skates	-\$8	paid to Omar
Omar	Earnings	\$8	received from Sam

Check final value:

Sam: $10 + 20 - 8 = 22 \geq 17$ (fair share for Sam)

Omar: $15 + 8 = 23 \geq 18$ (fair share for Omar)

Same Example with Different Bids: The Bids

	Sam	Omar
vase	\$10	\$5
earrings	\$4	\$1
ice skates	\$20	\$10
total bid	\$34	\$16
fair share	$\frac{34}{2} = \$17$	$\frac{16}{2} = \$8$

Same Example with Different Bids: The Awards

	Sam	Omar
<i>fair share (from above)</i>	<i>\$17</i>	<i>\$8</i>
who gets what (award)	<i>vase earrings skates</i>	<i>nothing</i>
value of award	<i>\$34</i>	<i>\$0</i>
award value — fair share	<i>$34 - 17 = 17$</i>	<i>$0 - 8 = -8$</i>
pays in / receives	<i>pays in \$17</i>	<i>receives \$8 from pot</i>

vase	\$10	\$5
earrings	\$4	\$1
ice skates	\$20	\$10

Same Example with Different Bids: The Surplus

	Sam	Omar
<i>pays in / receives (from above)</i>	<i>pays \$17</i>	<i>receives \$8</i>
total surplus	<i>$\underset{\text{in}}{17} - \underset{\text{out}}{8} = 9$</i>	
share of surplus	<i>$\frac{9}{2} = 4.50$</i>	<i>$\frac{9}{2} = 4.50$</i>
Final allocation	<i>Vase, earnings, skates, pay out \$17 getting \$4.50 from surplus</i>	<i>gets \$8 from pot \$4.50 from surplus</i>

Same Example with Different Bids: Summary

person	items	cash	paid or received?
Sam	Vase Skates earnings	paid \$17 got \$4.50 <u>pays</u> $17 - 4.50 = \$12.50$	\$12.50 paid to Omar
Omar	nothing	got \$8 got \$4.50 total \$12.50	\$12.50 received from Sam

Check: Sam's total value = $\$34 - 12.50 = \$21.50 \geq 17$
 Omar's total value = $\$0 + 12.50 = \$12.50 \geq 8$

Is the Outcome Fair?

Is the outcome fair to each person according to their values?

Yes! Everyone gets (stuff + cash) that is at least a fair share according to the \$ valuation placed on the items!

NO EMOTIONAL ATTACHMENT!

- What keeps someone from bidding high to get everything?
- Or bidding high even if they don't want something?