

1 Fair Division

1.1 Methods by Name

Divider-Chooser, Lone Divider, the Method of Sealed Bids.

1.2 Sample Fair Division Problems

- For each of the three methods listed above, what is the **context** in which they are appropriate? (for example: how many people are dividing? How can you split up the stuff?)

Divider-Chooser: only 2 people, continuous division (you can split stuff as finely as you want)

Lone Divider: more than 2 people, continuous division

Sealed Bids: for stuff (houses etc) that can't be subdivided

- The Cookie Monster and Elmo are splitting a giant cookie that is half chocolate-chip and half oatmeal, worth \$12. The Cookie Monster likes chocolate chip twice as much as oatmeal. Elmo likes oatmeal three times as much as chocolate chip.

- Suppose someone splits the cookie with all chocolate chip on one side and all oatmeal on the other. Attach dollar values to each share for each person.

	share 1: chocolate chip	share 2: oatmeal	<u>total</u>
Cookie Monster	8	4	12
Elmo	3	9	12

Proportion method for CM: guess oatmeal = 1, CC = 2(1) = 2. total = 3.
But we need the total to be 12, so compute $\frac{12}{3} = 4$ and multiply everything by 4.

Algebra method for Elmo: let c = value of chocolate chip. We need
 $c + 3c = 12 \Rightarrow 4c = 12 \Rightarrow c = 3$, oatmeal = $3(c) = 3 \cdot 3 = 9$

- Which cookie pieces (if any) are fair shares for Elmo? Which are fair shares for Cookie Monster? Show calculations that support your answer.

Fair share = $\frac{12}{2} = 6$. So S_1 is fair for CM

Since $8 > 6$. S_2 is fair for Elmo, since $9 > 6$.

3. (a) James, Gordon, Julie, Alexei, and Latrice divide their pile of Halloween candy worth a total of \$20. Determine who was the divider and determine how the lone divider method would proceed.

	Pile 1	Pile 2	Pile 3	Pile 4	Pile 5
James	\$0	\$5	\$10	\$5	\$0
Gordon	\$2	\$4	\$6	\$2	\$6
Julie	\$4	\$4	\$4	\$4	\$4
Alexei	\$2	\$4	\$12	\$2	\$0
Latrice	\$2	\$10	\$7	\$1	\$0

$$\text{Fair share} = \frac{\text{Value}}{\# \text{ people}} = \frac{20}{5} = 4.$$

Notice Julie values each of the piles at \$4 so she was the divider

Fair shares for each person are highlighted

It is possible to allocate shares so that everyone is happy, see blue circles.

- (b) Suppose **Gordon** changes his evaluation of the piles. Now determine how the lone divider method would proceed.

	Pile 1	Pile 2	Pile 3	Pile 4	Pile 5
James	\$0	\$5	\$10	\$5	\$0
Gordon	\$2	\$7	\$6	\$2	\$3
Julie	\$4	\$4	\$4	\$4	\$4
Alexei	\$2	\$4	\$12	\$2	\$0
Latrice	\$2	\$10	\$7	\$1	\$0

It is not possible to allocate shares to make everyone happy. We assign Julie an uncontested share (either share 1 or share 5) and then recombine the remaining shares.

Those recombined shares will be assigned to a new Lone Divider (chosen from James, Gordon, Alexei, Latrice) who will do the process again.

4. Harry Potter (P), Hermione Granger (G), Luna Lovegood (L), and Ronald Weasley (W) are dividing up some loot they found in the dungeons. The loot consists of a velvet cloak, a gold chalice, and a self-cleaning cauldron. They decide to divide the loot using the sealed bid method. The table below shows how many gold coins each person bid for each item.

(a) Fill in the following table.

	Potter	Granger	Lovegood	Weasley
cloak	60 coins	50 coins	100 coins	30 coins
chalice	60 coins	10 coins	0 coins	20 coins
cauldron	0 coins	80 coins	100 coins	30 coins
total bid	120	140	200	80
What is a fair share for each person?	$\frac{120}{4} = 30$	$\frac{140}{4} = 35$	50	20
award (who gets what item)	chalice	nothing	cloak, cauldron	nothing
award value	60	0	200	0
How much money does each person pay in or receive from the holding pile? <i>Award - fair share</i>	$60 - 30 = 30$ PAY	$0 - 35 = -35$ GET	$200 - 50 = 150$ PAY	$0 - 20 = -20$ GET
Determine the surplus.	$(\text{PAY}) - (\text{GET}) = 30 + 150 - 35 - 20 = 125$			
Determine each person's fair share of the surplus.	31.25	31.25	31.25	31.25

$$4 \times 31.25 = 125$$

- (b) Determine the Final Allocation by filling out each blank below. Space is provided here to determine the total cash for each person.

POTTER: $-30 + 31.25 = 1.25$

GRANGER: $35 + 31.25 = 66.25$

LOVEGOOD: $-150 + 31.25 = -118.75$

WEASLEY: $20 + 31.25 = 51.25$

Summary of Final Allocation			
person	items	cash	paid or received?
Potter	Chalice	1.25	received
Granger	—	66.25	received
Lovegood	cloak, cauldron	118.75	paid
Weasley	—	51.25	received

- (c) Is this a fair allocation? Write a computation to explain how you know.

cash check: $\underbrace{1.25 + 66.25 + 51.25}_{\text{received}} = \underbrace{118.75}_{\text{paid}}$

Fair? Total Values

$$\begin{array}{l|l} \text{P} & 60 + 1.25 = 61.25 > 30 \\ \text{G} & 66.25 > 35 \\ \text{L} & 200 - 118.75 = 81.25 > 50 \\ \text{W} & 51.25 > 20 \end{array}$$

everyone got stuff / cash whose total value is more than their fair share.

2 Graph Theory

1. Define the terms below.

(a) An **Euler circuit** is this defines "circuit"

A circuit (sequence of vertices and edges in a graph that starts and ends at the same vertex) that uses every edge in a graph exactly once

(b) An **Euler path** is this defines "path"

A sequence of vertices & edges in a graph that uses every edge exactly once. Typically the start & end vertex are different

(c) A **Hamiltonian circuit** is

A circuit that passes through every vertex exactly once

(d) A **spanning tree** is

A collection of vertices & edges in a graph without any circuits ^{"tree"}
with a path between any pair of vertices so that every vertex shows up as the endpoint of one of the edges

(e) A **minimum cost spanning tree** is

A spanning tree of smallest weight

(trees are connected & have no circuits)

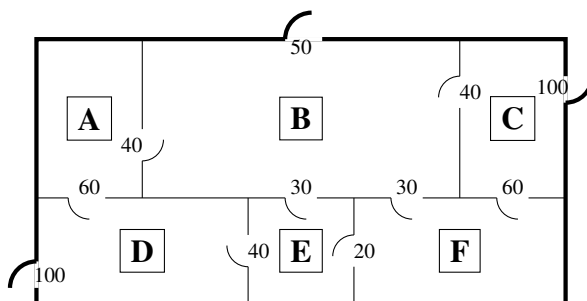
(f) the **degree of a vertex** is

the number of edges leaving each vertex
(both ends of loops are counted)

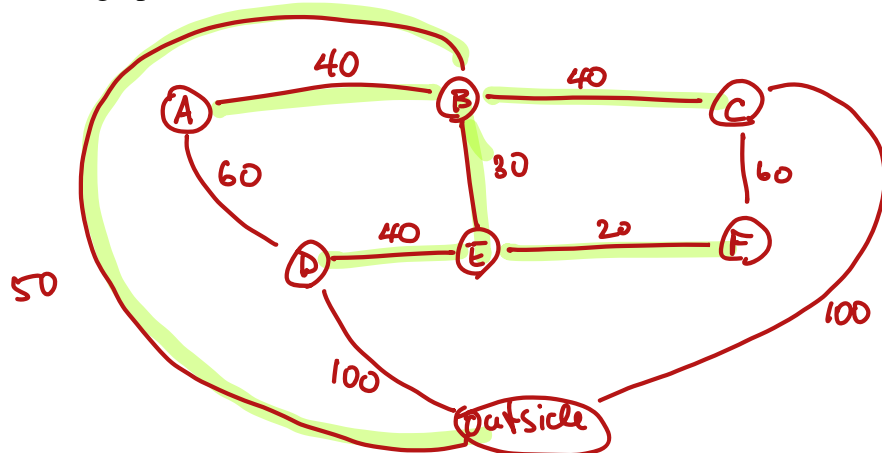
(g) A **complete graph** is

A graph that has an edge connecting every pair of vertices.

2. Below is the floor plan of an art gallery with 6 galleries and 11 doorways. A security company has assigned a theft risk assessment for each doorway. The higher the number, the greater the risk. Each gallery is labeled with a letter.



- (a) Draw a graph below that models this scenario.



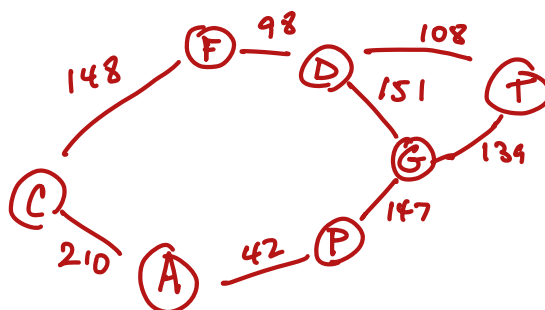
- (b) The gallery owners want all rooms to be available to visitors but they would like to shut and lock as many doors as possible to lower their risk. Using terminology from **Graph Theory**, what is the company looking for?

A minimum cost spanning tree! (Highlighted above)

You can get to every room, because the MCST includes every vertex. But the doors corresponding to expensive edges can be locked.

3. (a) Given the following table of distances (in miles) between towns in Alaska, construct a weighted graph model that represents the situation. A – in the table means there is not a direct route between those two towns (or there is a route but it passes through some other town on the list).

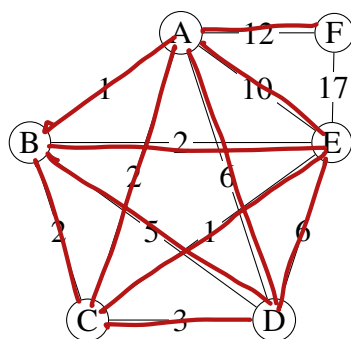
	Anchorage	Cantwell	Delta Junction	Fairbanks	Glennallen	Palmer	Tok
Anchorage	–	210	–	–	–	42	–
Cantwell	210	–	–	148	–	–	–
Delta Junction	–	–	–	98	151	–	108
Fairbanks	–	148	98	–	–	–	–
Glennallen	–	–	151	–	–	147	139
Palmer	42	–	–	–	147	–	–
Tok	–	–	108	–	139	–	–



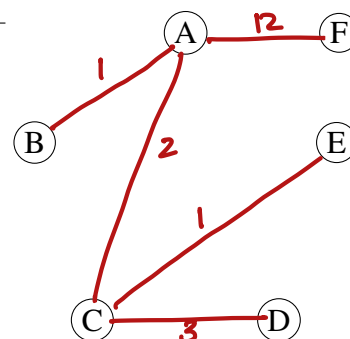
- (b) Is the graph connected? Why or why not? Use terminology from graph theory to explain your answer.

Yes, there is a path between any pair of vertices.

4. (a) Use **Kruskal's Algorithm** to find a **minimum cost spanning tree** in the graph below. Document each edge you looked at, in order, and whether or not that edge is used in the MCST.



edge	weight	used?
AB	1	✓
CE	1	✓
AC	2	✓
BC	2	x
BE	2	x
CD	3	✓
BD	5	x
AD	6	x
DE	6	x
AE	10	x
AF	12	✓ STOP



- (b) Give an example of a real-world problem where you would want to find a minimum weight spanning tree. (You need to explain what the vertices represent, what the edges represent, and what the weights represent, and how a minimum weight spanning tree is useful.)

hooking up houses to natural gas, & you want to minimize the total cost of laying the pipe!

Vertices are houses

Edges are possible routes to lay pipe

weights are costs associated with each pipe route

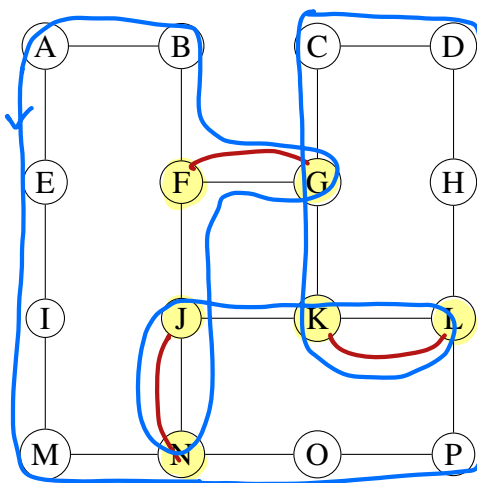
5. (a) What does it mean to **Eulerize** a graph? What is your goal? What kind of edges can you add?

You want to duplicate existing edges so that every vertex has even degree. This will allow the construction of an Euler circuit.

- (b) **Eulerize** the graph below by duplicating the fewest number of edges and then find an **Euler circuit** in the resulting graph. Make sure that someone else can clearly see the additional edges. **List** the additional edges and the vertices in the order they are traversed in the Euler circuit.

Offset your circuit slightly from the graph so that it is clear where the circuit is, and draw it as one **continuous** path. If possible use a different color.

odd degree vtx are highlighted.



added edges: GF, JN, KL

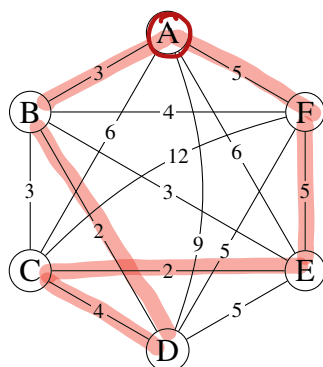
Euler Circuit: A E Z M N O P L H D C G K L K J N J F G F B A

- (c) Give an example of a real-world problem where you would want to find an Euler circuit. (You need to explain what the vertices represent, what the edges represent, what the weights represent, and why finding an Euler circuit is useful in answering the problem.)

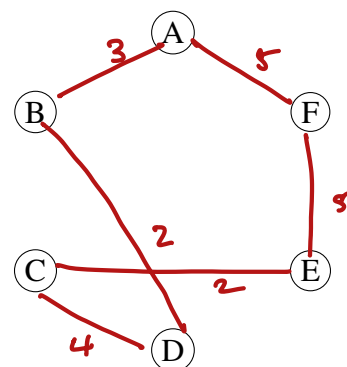
Snow plowing! Vertices are intersections, edges are roads, weights are length of road segment

and you want to travel on every road in the neighborhood repeating the shortest possible distance (travelling the least on already plowed roads)

6. (a) Use the **Nearest Neighbor Algorithm** starting at vertex A, to find a **Hamiltonian Circuit** in the graph below and determine its weight. List the edges in the table in the order you considered them / added them.



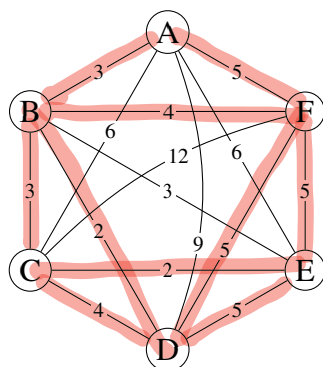
edge	weight	used?
AB	3	Yes
BC	2	Yes
CD	4	Yes
CE	2	Yes
EF	5	Yes
AF	5	Yes



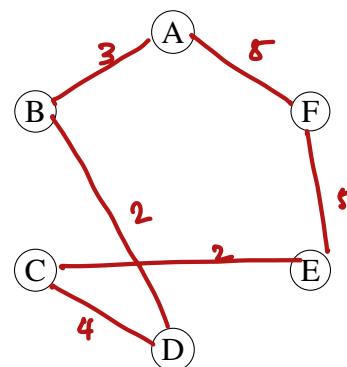
circuit: ABDCEFA

weight: 21

- (b) Use the **Cheapest Link Algorithm** to find a **Hamiltonian Circuit** in the graph below, and determine its weight. List the edges in the table in the order you considered them / added them.



edge	weight	used?
BD	2	✓
CE	2	✓
AB	3	✓
BC	3	x
BF	4	x
CD	4	✓
AF	5	✓
DE	5	x
DF	5	x
FE	5	✓



circuit: ABDCEFA (same as above, that can happen!)

weight: 21

- (c) Give an example of a real-world problem where you would want to find a minimum weight Hamiltonian circuit. (You need to explain what the vertices represent, what the edges represent, and what the weights represent, and why finding a Hamiltonian circuit is useful in answering the problem.)

Travelling Salesman Problem: visit a collection of cities as cheaply as possible and return home

Vertices are cities

edges are possible routes

weights are cost of route (driving time, airline ticket cost, etc.)