

Note that before Fall 2026, Exam 1 covered Voting Theory, Weighted Voting, and Fair Division. This Exam 1 review material covers Voting Theory and Weighted Voting only.

# Voting Theory

## 1.1 You should know

**Terminology:** majority, plurality, Cordorcet Winner

**Voting Methods by Name:** Plurality Method, Instant Run-off Voting (IRV) (also known as Ranked Choice Voting), Borda Count, Copeland's Method

**Fairness Criteria by Name:** Condorcet Criterion, Monotonicity Criterion, Majority Criterion, Irrelevance of Independent Alternatives (IIA) Criterion

## 1.2 Sample Voting Theory Problems

1. A class of middle schoolers are trying to decide what would be the worst ingredient to add to a cake their Principal has to eat if they win the Science Bowl. They narrow their options down to three: pickled onions (PO), stinky cheese (SC), or anchovies (A). The class is polled and the resulting preference schedule is below.

|            | 22 | 5             | 23            | 2             | 11            | 20            |
|------------|----|---------------|---------------|---------------|---------------|---------------|
| 1st choice | PO | PO            | <del>SC</del> | <del>SC</del> | A             | A             |
| 2nd choice | SC | A             | PO            | A             | PO            | <del>SC</del> |
| 3rd choice | A  | <del>SC</del> | A             | PO            | <del>SC</del> | PO            |

- (a) How many students voted? 83  
 $22 + 5 + 23 + 2 + 11 + 20 = 83$
- (b) How many votes does a candidate need to win in order to win a majority? Show the calculation that gives your answer. 42  
 $83/2 = 41.5$
- (c) How many votes does a candidate need to win in order to win a plurality? Show the calculation that gives your answer. 28  
 $83/3 \approx 27.7$  so they need to win at least 28
- (d) Find the winner using the plurality method. Show your work.  
 $PO = 22 + 5 = 27$   
 $SC = 23 + 2 = 25$   
 $A = 11 + 20 = 31$  ← largest  
A wins w/ 31 votes
- (e) Find the winner using Instant Runoff Voting. Show your work.

Round 1  
 $A = 31$   
 $PO = 27$   
 $SC = 25$   
SC is eliminated

Round 2  
 $A = 31 + 2 = 33$   
 $PO = 27 + 23 = 50$  ← The winner under IRV!  
check  $50 + 33 = 83$  ✓

For reference, here's the preference schedule again.

|     |            |    |    |    |    |    |    |
|-----|------------|----|----|----|----|----|----|
|     |            | 22 | 5  | 23 | 2  | 11 | 20 |
| pts |            |    |    |    |    |    |    |
| 3   | 1st choice | PO | PO | SC | SC | A  | A  |
| 2   | 2nd choice | SC | A  | PO | A  | PO | SC |
| 1   | 3rd choice | A  | SC | A  | PO | SC | PO |

(f) Find the winner using Borda Count. Show your work.

$$\begin{aligned}
 A &= 22(1) + 5(2) + 23(1) + 2(2) + 11(3) + 20(3) = 152 \\
 SC &= 22(2) + 5(1) + 23(3) + 2(3) + 11(1) + 20(2) = \boxed{175} \leftarrow \text{the most} \\
 PO &= 22(3) + 5(3) + 23(2) + 2(1) + 11(2) + 20(1) = 171
 \end{aligned}$$

SC wins because it got the most points.

(g) Find the winner using Copeland's Method. Show your work.

A vs PO

|            | 22            | 5             | 23            | 2             | 11            | 20            |
|------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 1st choice | PO            | PO            | <del>SC</del> | <del>SC</del> | A             | A             |
| 2nd choice | <del>SC</del> | A             | PO            | A             | PO            | <del>SC</del> |
| 3rd choice | A             | <del>SC</del> | A             | PO            | <del>SC</del> | PO            |

$$PO = 22 + 5 + 23 = 50$$

$$A = 2 + 11 + 20 = 33$$

PO wins

A vs SC

|            | 22            | 5             | 23            | 2             | 11            | 20            |
|------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 1st choice | <del>PO</del> | <del>PO</del> | SC            | SC            | A             | A             |
| 2nd choice | SC            | A             | <del>PO</del> | A             | <del>PO</del> | SC            |
| 3rd choice | A             | SC            | A             | <del>PO</del> | SC            | <del>PO</del> |

$$A = 5 + 11 + 20 = 36$$

$$SC = 22 + 23 + 2 = 47$$

SC wins

PO vs SC

|            | 22           | 5            | 23            | 2            | 11           | 20           |
|------------|--------------|--------------|---------------|--------------|--------------|--------------|
| 1st choice | PO           | PO           | SC            | SC           | <del>A</del> | <del>A</del> |
| 2nd choice | SC           | <del>A</del> | <del>PO</del> | <del>A</del> | PO           | SC           |
| 3rd choice | <del>A</del> | SC           | <del>A</del>  | PO           | SC           | PO           |

$$PO = 22 + 5 + 11 = 38$$

$$SC = 23 + 2 + 20 = 45$$

SC wins

| Candidate | Points |
|-----------|--------|
| A         | 0      |
| PO        | 1      |
| SC        | 2      |

SC wins under Copeland's method

(h) Is any candidate a Condorcet Winner? Explain your answer.

A Condorcet winner needs to win all their head-to-head races. Observe that SC won all its head-to-head races (SC vs A, SC vs PO) so it is a Condorcet winner.

2. Consider the following preference schedule, for an election with five candidates: Alfred, Bethanne, Carol, David, and Edgar.

| Rank | 40 | 8 | 25 | 15 | 5 | 7 |
|------|----|---|----|----|---|---|
| 1st  | A  | B | C  | D  | E | C |
| 2nd  | C  | A | A  | A  | A | B |
| 3rd  | B  | C | B  | B  | B | A |
| 4th  | D  | D | D  | C  | C | D |
| 5th  | E  | E | E  | E  | D | E |

$\leftarrow \text{total} = 40 + 8 + 25 + 15 + 5 + 7 = 100$

Need 51 for a majority

Find the winner under Instant Runoff Voting. Clearly show your rounds, and state who is eliminated in each round. For each round, clearly indicate the vote total for each candidate.

Round 1

$A = 40$

$B = 8$

$C = 25 + 7 = 32$

$D = 15$

$E = 5$

No majority.

E is eliminated

Round 2

$A = 40 + 5 = 45$

$B = 8$

$C = 32$

$D = 15$

No majority.

B is eliminated

Round 3

$A = 45 + 8 = 53$

$C = 32$

$D = 15$

A wins, with a majority!

The winner is A and they won in 3 rounds.

3. Consider the following preference schedule, and the points assigned under Borda count. Explain which fairness criterion this shows Borda Count fails, and why this particular election fails that fairness criterion.

| Rank | 51 | 30 | 19 | Candidate | Borda Count Points      |
|------|----|----|----|-----------|-------------------------|
| 1st  | A  | B  | C  | A         | 253                     |
| 2nd  | B  | C  | D  | B         | 311 $\leftarrow$ Winner |
| 3rd  | C  | D  | B  | C         | 268                     |
| 4th  | D  | A  | A  | D         | 168                     |

total votes = 100

A has a majority of 1<sup>st</sup> place votes but B wins under Borda Count. Fails the Majority condition.

4. For each of the following fairness criteria, describe what an election would look like that **failed** that criterion. (You do not need to provide an actual preference schedule, just talk about what the problem would be.)

(a) Condorcet Criterion

The winner loses at least one head-to-head race

(b) Monotonicity Criterion

If you re-run the election to assign more 1<sup>st</sup> choice votes to the winner, that causes the winner to lose.

(c) Majority Criterion

A candidate won a majority of the 1<sup>st</sup> choice votes but still lost the election

(d) Irrelevance of Independent Alternatives (IIA) Criterion

If you remove a losing candidate that changes the outcome of the election

## 2 Weighted Voting

### 2.1 Terminology

quota, weight, winning coalition, critical player in a winning coalition, dictator, veto power, dummy, Banzhaf Power Index or Banzhaf Power Distribution.

### 2.2 Sample Weighted Voting Problems

1. Nova Robotics Inc. has five board members who vote on major company decisions (funding rounds, executive hires, acquisitions, etc.). Each member's voting weight is based on the percentage of company shares they control, and company bylaws require any motion to pass with at least 60% of the total votes.

Founder/CEO: 56 shares

Co-founder/CTO: 44 shares

Lead Venture Capital firm: 40 shares

Early Angel Investor: 36 shares

Employee Stock Pool Representative: 24 shares

- (a) Write a weighted voting system of the form  $[q : w_1, \dots, w_n]$  that represents this scenario.

*Total weight =  $56 + 44 + 40 + 36 + 24 = 200$  and  $200(0.6) = 120$*

*$[120 : 56, 44, 40, 36, 24]$*

- (b) Find a winning coalition in this system, and write a sentence explaining how you know it is a winning coalition.

*Need a coalition that can make quota.*

*For example,  $\{P_1, P_2, P_3\}$  have weight*

*$56 + 44 + 40 = 140 > 120$  so they make quota if they vote together.*

2. For each weighted voting system below, determine if there are any dictators, anyone with veto power, or any dummies.

(a)  $[16 : 16, 11, 3, 1]$

Yes,  $P_1$  is a dictator since they can make quota alone

$11 + 3 + 1 = 15$  so everyone else is a dummy

(only  $P_1$  has veto power)

(b)  $[51 : 40, 30, 20, 10]$

No dictators since no one's quota is  $\geq 51$

$30 + 20 + 10 = 60$   $\leftarrow P_1$  does not have veto power

$40 + 20 + 10 = 70$   $\leftarrow P_2$  does not have veto power

$40 + 30 + 10 = 80$   $\leftarrow P_3$  does not have veto power

$40 + 30 + 20 = 90$   $\leftarrow P_4$  does not have veto power

No one is a dummy: for example

$P_4$  is critical in  $\{P_2, P_3, P_4\}$

(c)  $[31 : 10, 9, 8, 7, 6]$

No dictator since no one can make quota alone

$9 + 8 + 7 + 6 = 30$  so  $P_1$  has veto power

$10 + 8 + 7 + 6 = 31$  and everyone in  $\{P_1, P_3, P_4, P_5\}$  is critical,

so no dummies

3. Consider the weighted voting system  $[q : 21, 20, 7, 5]$  where the players can pass a motion with a **majority**.

(a) What is  $q$  in this case? How do you know?

total weight =  $21 + 20 + 7 + 5 = 53$      $\frac{53}{2} = 26.5$  so majority =  $q = \underline{27}$

(b) List all winning coalitions.

$P_1$ ,  $P_2$     sum = 31

$P_1$ ,  $P_3$     sum = 28

$P_2$ ,  $P_3$     sum = 27

$P_1$   $P_2$   $P_3$      $\leftarrow$  drop any one of  $P_1, P_2, P_3$ , still winning.

$P_1$   $P_2$   $P_4$

$P_1$   $P_3$   $P_4$

$P_2$   $P_3$   $P_4$

$P_1$   $P_2$   $P_3$   $P_4$      $\leftarrow$  Drop any single player & still make quota

(c) In each of the winning coalitions above, indicate who is a critical player. *by underlining.*

(d) Calculate the Banzhaf Power Index for each player.

|       | <u># underlines</u> |      |     |
|-------|---------------------|------|-----|
| $P_1$ | <u>4</u>            | 4/12 | 33% |
| $P_2$ | <u>4</u>            | 4/12 | 33% |
| $P_3$ | <u>4</u>            | 4/12 | 33% |
| $P_4$ | 0                   | 0/12 | 0%  |
| total | 12                  |      |     |

(rounded)

(e) Based on your calculations above, are there any dummies? Explain.

Yes,  $P_4$  is a dummy. It is never critical / has no power.

(Observe everyone else has equal power despite different #s of votes!)