Some Solutions

1. In earlier notes, we found seven possible coalitions with players $P_1$, $P_2$, and $P_3$.

   (a) List them again below.

   $P_1$
   $P_2$
   $P_3$
   $P_1, P_2$
   $P_1, P_3$
   $P_2, P_3$
   $P_1, P_2, P_3$

   total number: 7

   (b) Suppose the system has a fourth player, $P_4$. Determine how many coalitions in this case. Try to answer the question without actually listing all of them.

   Observation 1: All of the coalitions above are also coalitions with players $P_1$, $P_2$, $P_3$, and $P_4$. So we get the 7 above.

   Observation 2: Every time we add $P_4$ to a coalition above, we get another one. For example, adding $P_4$ to the first three in the list gives:

   $P_1, P_4$
   $P_2, P_4$
   $P_3, P_4$

   Observation 3: We missed out coalition $P_4$

   total number: $7+7+1=15$

   (c) What if there is a fifth player, $P_5$?

   guess: $15+15+1=31$

   (d) Make a conjecture about how many coalitions are possible with $n$ players, $P_1, P_2, P_3, \cdots P_n$. How would you argue that your count is correct?

   What is the pattern you see?
   $7 = 8-1 = 2^3-1$
   $15 = 16-1 = 2^4-1$
   $31 = 32-1 = 2^5-1$
   guess: With $n$ players there are $2^n - 1$ different coalitions.

(e) What does this suggest about the mechanics of calculating the Banzhaf Power Index for a weighted voting system with a lot of players?

$2^n$ grows really really fast.
While $2^5 = 32$, $2^{10} = 1024$, and $2^{20} > 1,000,000$.
Listing **all** coalitions is, in general, not practical.

2. What is the Banzhaf Power Index

(a) when there is a dictator

dictator has 100%, all others have 0%

(b) for a dummy player

always has 0%

(c) if all players have an equal number of votes

All players have equal power. Might has well have 1 player 1vote.

(d) if player $P_1$ has double the number of votes as player $P_2$?

All you can conclude is that $P_2$ will not have **more** power than $P_1$. But beyond that there isn't enough information. We have seen examples in which such players have equal power and we can construct others where $P_1$ is a dictator: $[10 : 10, 5, 2]$