Worksheet 5: Weighted Voting and Banzhaf Power Index

1. The weighted voting system, $[12 : 11, 8, 4]$, assigned voting weights to players P_1, P_2, and P_3 according to how many hours per week they worked at a co-owned business.

   (a) Given this system, what is your *intuition* regarding the distribution of power among the players? Make a rough estimate for how you think the power is distributed. (This means assigning a percentage to each player such that the numbers sum to 100%.)

   Many correct answers.
   Since 8 is twice 4, and 11 is almost 3×4, maybe P_1 : 60%, P_2 : 40%, P_3 : 20% ?

   (b) List all possible winning coalitions. (Hint: There are 4.)

   P_1, P_2 : $11+8=19 \geq 12$
   P_2, P_3 : $8+4=12 \geq 12$
   P_1, P_3 : $11+4=15 \geq 12$
   P_1, P_2, P_3 : $11+8+4=23 \geq 12$

   (c) Calculate the Banzhaf power distribution.

   total # critical = 6 ;

Player	# player's critical	$\frac{\#\text{ player's critical}}{\text{total critical}}$
P_1	2	$2/6 = 33.3\%$
P_2	2	$2/6 = 33.3\%$
P_3	2	$2/6 = 33.3\%$

   (d) Are you surprised? Do you think this system is operating as the players intended?

   All have equal power. No. I think they intended players with more votes to have more power.

   (e) Explain why the quota cannot be any smaller than 12 or any larger than 23.

   total weight is 23; So $q \leq 23$.
   and $23/2 = 11.5$; So 12 is the smallest.

   (f) Re-calculate the Banzhaf power distribution with the quota changed from 12 to 13.

   Only winning coalitions P_1, P_2 and P_1, P_3 and P_1, P_2, P_3
   P_1 : $\frac{2}{4} = 50\%$
   P_2 : $\frac{1}{4} = 25\%$
   P_3 : $\frac{1}{4} = 25\%$

   (g) What other changes to this voting system could shift the Banzhaf power distribution?

   Give some players more votes and give others fewer votes.
   Say $[12 : 12, 10, 2]$.

2. A condo community is voting to approve a \$10K loan to fix structural issues with the building. If passed, the homeowners are responsible for paying off the loan over the next five years which would result in an increase in the homeowners' monthly HOA fee. There are 2 homeowners (H_1 and H_2) and 2 board members (B_1 and B_2) on the committee that will vote on the loan proposal.

   For this measure to pass both homeowners and at least one board member must vote yes.

   (a) Construct a weighted voting system that represents this setup. You will need to identify which weights are associated with the homeowners (H_1 and H_2) and which are associated with the board members, (B_1 and B_2).

   (You may need to try several options! See the next question to determine if your system works.)

   [5 : 2, 2, 1, 1]
   H_1 H_2 B_1 B_2

   (b) Show your system works, by verifying the following two conditions:

      i. Every coalition of the form HHB or HHBB makes quota

      $H_1H_2B_1$: $2+2+1=5 \geq 5$

      $H_1H_2B_2$: $2+2+1=5 \geq 5$

      $H_1H_2B_1B_2$: $2+2+1+1=6 \geq 5$

      ii. Every other coalition does NOT make quota

      $H_1B_1B_2$ or $H_2B_1B_2$: $2+1+1=4<5$

      H_1 or H_2 : $2<5$

      B_1 or B_2 : $1<5$